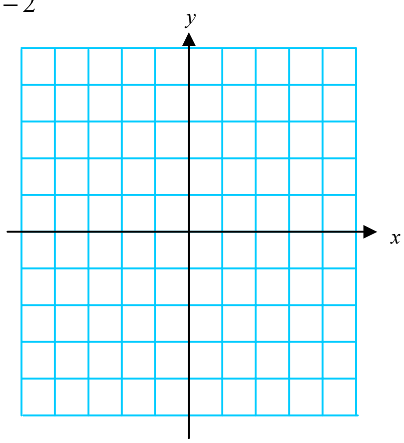
1.2 FINDING LIMITS GRAPHICALLY AND NUMERICALLY

An Introduction to Limits

Example: Sketch the graph of $f(x)=\dfrac{x^2-4}{x-2}$; $x \neq 2$



(a) What happens at $x = 2$?

(b) Even though this function is not defined at $x = 2$, we can still examine its behavior close to $x = 2$. Complete the table of values of the function close to $x = 2$.

x approaches 2 from the **left** → | ← x approaches 2 from the **right**

x	1.5	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25	2.5
$f(x)$											

Informal Definition of a Limit

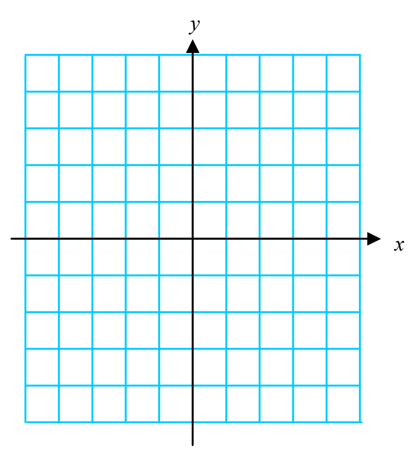
Suppose a function f is defined on an interval around c, but possibly not at the point $x = c$ itself. Suppose that as x becomes sufficiently close to c, $f(x)$ becomes as close to a single number L as we please.
We then say that the **limit of $f(x)$ as x approaches c is L**, and we write

$$\lim_{x\to c} f(x) = L$$

(c) Apply this definition to the function from above to find the $\lim_{x\to 2} f(x)$.

Example: Find $\lim_{x\to 2} g(x)$, where g is defined as

$$g(x)=\begin{cases}1, & x\neq 2\\ 0, & x=2\end{cases}$$



There are 3 basic ways to evaluate a limit. So far we have used two of them. In section 3, we will use the third.

1. Numerical approach … Make a table
2. Graphical approach … Draw a graph by hand or using the calculator
3. Analytical approach … Use algebra or calculus (section 3)

The Formal Definition of a Limit

When we say "$f(x)$ becomes as close to L as we please" in the informal definition, we mean that we can specify a maximum distance between $f(x)$ and L. This distance is given by

$$|f(x)-L| = \text{Distance between } f(x) \text{ and } L.$$

We use the Greek letter ε (epsilon) to stand for the maximum distance, so we require

$$|f(x)-L| < \varepsilon .$$

Similarly, we interpret "x becomes sufficiently close to c" to mean

$$|x-c| < \delta ,$$

where the Greek letter δ (delta) tells us how close x must be to c. Then

$$\lim_{x\to c} f(x) = L$$

means that we can make the distance $|f(x)-L|$ between the function values and L as small as we like (less than any number $\varepsilon > 0$) by making the distance $|x-c|$ between x and c sufficiently small (less than some $\delta > 0$).

Definition: Suppose a function f is defined on an interval around c, but possibly not at the point $x = c$ itself. Suppose that for any $\varepsilon > 0$ (as small as we like), there is a $\delta > 0$ (sufficiently small) so that

$$\text{if } 0 < |x-c| < \delta \text{ , then } |f(x)-L| < \varepsilon .$$

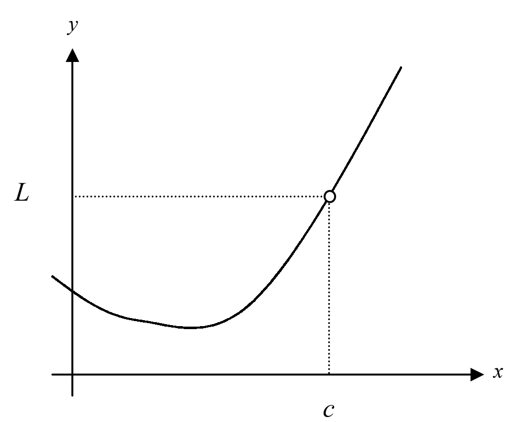
We then say the limit of $f(x)$ as x approaches c exists and write

$$\lim_{x\to c} f(x) = L .$$

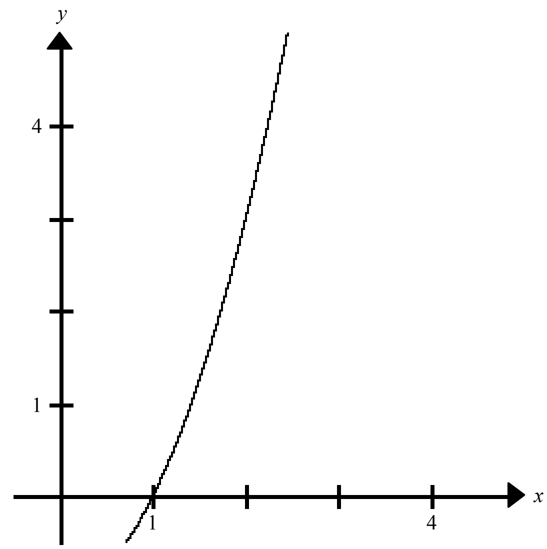
(Just for fun ☺) Symbolically this can be written as follows:

$$\lim_{x\to c} f(x) = L \Leftrightarrow (\forall \varepsilon > 0)(\exists \delta > 0) \ni (0 < |x-c| < \delta) \Rightarrow (|f(x)-L| < \varepsilon)$$

Example: Consider the following function. Graphically show the definition of a limit.



Example: The graph of $f(x) = x^2 - 1$ is shown below. Find δ such that if $0 < |x-2| < \delta$, then $|f(x) - 3| < 0.2$



When Limits Do Not Exist

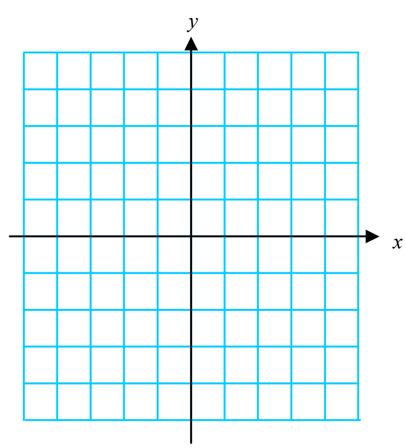
If there does not exist a number L satisfying the condition in the definition, then we say the $\lim_{x \to c} f(x)$ does not exist.

Limits typically fail for three reasons:

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

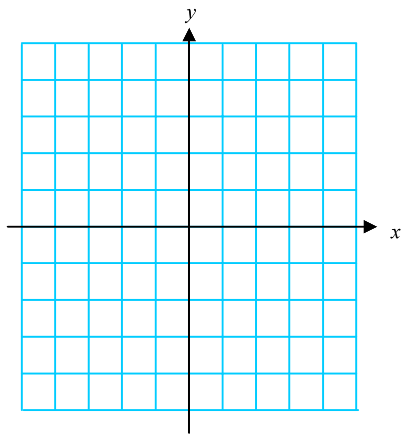
Example: Investigate the existence of the following limits.

(a) $\lim_{x \to 0} \frac{|x|}{x}$



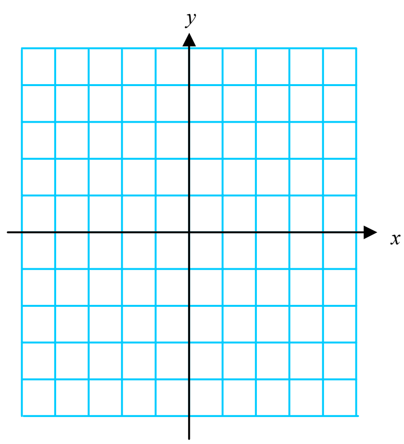
x	–0.5	–0.25	–0.1	–.01	–.001	0	.001	.01	.1	.25	.5
$f(x)$											

(b) $\lim_{x \to 1} \frac{1}{(x-1)^2}$



x	0	.5	.9	.99	.999	1	1.001	1.01	1.1	1.5	2
$f(x)$											

(c) $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$



x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$\frac{2}{13\pi}$	As $x \to 0$
$f(x)$								

♬ **(NOTE)**: Throughout the text, when you see

$$\lim_{x \to c} f(x) = L,$$

two statements are *implied* → (1) the limit exists, and (2) the limit is L.