Introduction to Limits of Functions

Goals

- To develop an intuitive understanding of the nature of limits.
- To lay the foundation for the frequent use of limits in calculus.
- To experience the power and peril of investigating limits by successively closer evaluation.

In the Lab

In this lab we shall study the behavior of a function f near a specified point. While this is sometimes a straightforward process, it can also be quite subtle; in many instances in calculus the process for finding a limit must be applied carefully. By gaining an intuitive feel for the notion of limits, you will be laying a solid foundation for success in calculus.

1. Consider the function f defined by $f(x) = \frac{x^4-1}{x-1}$.
   a. Complete the following table of values:

x	1.8	1.9	1.99	1.999	1.9999
$f(x)$					

   What do you think happens to the values of f as x increases towards 2?

   b. Complete the following table of values:

x	2.2	2.1	2.01	2.001	2.0001
$f(x)$					

   What do you think happens to the values of f as x decreases towards 2?

   As a shorthand we shall describe what you found in parts *a* and *b* by writing $\lim_{x\to 2} f(x) = 15$, or more specifically,

$$\lim_{x\to 2} \frac{x^4-1}{x-1} = 15.$$

c. In this particular case you could have "cheated" by immediately evaluating f at 2. Use your calculator to get a plot of the function between 1.8 and 2.2 to illustrate what happens in this straightforward situation. Draw a sketch of your plot here.

2. Use the same function f as above, but this time consider what happens as x approaches 1.

a. Complete the following table of values:

x	0.8	0.9	0.99	0.999	0.9999
$f(x)$					

What do you think happens to the values of f as x increases towards 1?

b. Complete the following table of values:

x	1.2	1.1	1.01	1.001	1.0001
$f(x)$					

What do you think happens to the values of f as x decreases towards 1?

Now consider the numerator and denominator separately to gain some additional feel and respect for the situation.

c. Complete the following table of values:

x	0.8	0.9	0.99	0.999	0.9999
$x^4 - 1$					
$x - 1$					

What do you think happens to the values of the numerator as x increases towards 1?

What do you think happens to the values of the denominator as x increases towards 1?

d. Complete the following table of values:

x	1.2	1.1	1.01	1.001	1.0001
$x^4 - 1$					
$x - 1$					

What do you think happens to the values of the numerator as x decreases towards 1?

What do you think happens to the values of the denominator as x decreases towards 1?

What are your conclusions and, in particular, what is $\lim_{x \to 1} f(x)$?

e. What happens when you try to "cheat" as was done in part *c* of Problem 1?

There are situations in which direct evaluation at the specified point is possible and actually gives the limit. These give rise to a concept called *continuity*. There are many important situations in calculus when this technique will not work, however.

3. Try to determine the values of the following limits. Use either graphing or function evaluation at nearby points. Show the method you used to determine your answers.

   a. $\lim_{x\to 0}\frac{\sin(10x)}{x}$

   b. $\lim_{x\to 1} g(x)$ where

$$g(x) = \begin{cases} \dfrac{x^4-1}{x-1}, & \text{for } x < 1, \\ 17, & \text{for } x = 1, \\ 14 - \dfrac{10}{x}, & \text{for } x > 1 \end{cases}$$

   Important Comment and Hint: As far as the existence and value of the limit are concerned, the value of $g(1)$ has no relevance.

   c. $\lim_{x\to 0}(1+x)^{1/x}$

   Record your best guess; the limit is a famous mathematical constant.

   d. $\lim_{t\to 1}\frac{t^n-1}{t-1}$ for a general positive integer n

   Hint: Recall Problem 2, try other values of n, and generalize.

4. It is important to be aware that limits can sometimes fail to exist. Investigate the following limits and explain why you think each does not exist. You may find it helpful to use the computer to evaluate the functions at different values of x and to plot graphs of the functions. Show the method you used to determine your answer.
   a. $\lim_{x\to 2}\frac{x}{x-2}$

   b. $\lim_{x\to 0}\frac{\sin(10x)}{x^2}$

   c. $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$

   d. $\lim_{x\to 0}\frac{|x|}{x}$

   e. $\lim_{x\to 4} f(x)$, where

$$f(x) = \begin{cases} (x+2)^3 & \text{for } x < 4 \\ e^x & \text{for } x > 4 \end{cases}$$

Note: For parts d and e think about the idea of a "one-sided limit" and store your thoughts for future reference. You should also convince yourself that computer evaluation is not really needed to make wise conclusions in these particular situations.

Further Exploration

5. Complete the following table by evaluating each function at the indicated x-values.

x	1	10	100	1000	10000
$f(x)=\dfrac{3x^4-x^2+10}{2x^4+\frac{5}{x}}$					
$g(x)=\dfrac{\sin x}{1+x^2}$					
$h(x)=\dfrac{4-\frac{3}{x}}{\sin x}$					

Based on the values found in the table, find the following limits, if they exist:

$\lim\limits_{x\to\infty} f(x) =$ $\qquad$ $\lim\limits_{x\to\infty} g(x) =$ $\qquad$ $\lim\limits_{x\to\infty} h(x) =$

Hints: The limit exists for two of these functions.

6. Try to determine the limit $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ when $f(x)=\ln x$.

   Hints: After specifying the quotient for the given f, treat x as a constant and give x a specific value of your own choosing before investigating the limit as h goes to 0. Then repeat for several other x values, make table of results, and try to see the pattern. What function of x emerges as you compute this limit for values of x? Limits of this particular *difference quotient* are very important in calculus.

Conclusions

What have you learned as a result of completing this lab that you did not know before?

- In your own words, what is meant by "limit"?
- When can you find the limit by "cheating" and substituting the value of x in the function?
- Under what conditions would a limit not exist?